

# Schwinger pair production: Explicit Localization of the world-line instanton

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## Abstract

We present a simple proof that the imaginary part of the world-line path integral which computes the rate of Schwinger pair production of charged particles in a constant electric field, is given exactly by the semiclassical WKB limit.

Schwinger's famous formula [1] computes the probability,  $P = 1 - e^{-\gamma V}$ , of the production of charged particle-antiparticle pairs by a constant external electric field, where, for spin zero particles,

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} E^2}{8\pi^3 n^2} e^{-\pi m^2 n/|E|} \quad (1)$$

Here  $m$  is the mass of the particles,  $V$  is the space-time volume and  $E$  is the electric field, into which we have absorbed a factor of the electric charge. The problem of computing the damping rate  $\gamma$  can be posed as that of evaluating the imaginary part of the world-line path integral for the relativistic particle,

$$\gamma = -2\Im \frac{1}{V} \int_0^\infty \frac{dT}{T} \int [dx_\mu(\tau)] e^{-\int_0^1 d\tau \left[ \frac{T}{4} \dot{x}_\mu(\tau) \dot{x}_\mu(\tau) - Ex_1(\tau) \dot{x}_2(\tau) \right] - \frac{m^2}{T}} \quad (2)$$

The integral is over periodic paths,  $x_\mu(\tau + 1) = x_\mu(\tau)$  and the space-time metric has Euclidean signature. Although the imaginary part of this integral can be readily be found to be (1) by integrating the Gaussian variables  $x_\mu(\tau)$  and then finding the imaginary part of the remaining integral over  $T$ , it is often convenient to perform a first principles semi-classical evaluation of the integral in (2). In that case, one treats  $x_\mu(\tau)$  and  $T$  as dynamical variables, solves the classical equations of motion, expands in fluctuations about the solution and integrates over the fluctuations. In fact, this second approach is needed for more general cases where the gauge field is not just a constant electric field [2] or when the gauge field is dynamical [3]. Then the action is

not just quadratic in the coordinates but has higher order terms and a perturbative expansion about the classical saddle points of the action is one possible approach. In the following, shall have nothing to say about the more general case. What we demonstrate is that the semiclassical approximation to (2) is not an approximation. It gives the exact result. It would be interesting to understand if this observation could be used to any advantage in the more general problems where the gauge field is non-constant or dynamical. We shall not address this interesting question in this paper.

An electric field in Euclidean space resembles a magnetic field and the classical instanton solutions are cyclotron orbits in that field [4]. The  $n$ -instanton solution is simply  $n$  repetitions of a single cyclotron orbit. The fluctuations of  $x_\mu(\tau)$  and  $T$  about those solutions have an odd number of tachyonic modes. As a result, the fluctuation determinants are negative and the instanton amplitudes, which contain the square roots of these determinants are imaginary and give the integral in (2), which is nominally an integral of a real integrand over real variables, an imaginary part.

The semiclassical computation treats both  $x_\mu(\tau)$  and  $T$  as dynamical variables which, to leading order, solve the classical equations of motion derived from the world-line action,

$$S = \int_0^1 d\tau \left[ \frac{T}{4} \dot{x}_\mu(\tau)^2 - E x_1(\tau) \dot{x}_2(\tau) + \frac{m^2}{T} \right] \quad (3)$$

The classical equations of motion are

$$\frac{1}{4m^2} \int_0^1 \dot{x}^2 = \frac{1}{T^2}, \quad -\frac{T}{2} \ddot{x}_1 - E \dot{x}_2 = 0, \quad -\frac{T}{2} \ddot{x}_2 + E \dot{x}_1 = 0, \quad -\frac{T}{2} \ddot{x}_{3,4} = 0 \quad (4)$$

with periodic boundary conditions,  $x_\mu(\tau + 1) = x_\mu(\tau)$ . The solutions of these equations are

$$x_{0\mu}^{(n)}(\tau) = \frac{m}{E} (\cos 2\pi n\tau, \sin 2\pi n\tau, 0, 0), \quad T_0^{(n)} = \frac{E}{\pi n}, \quad n = 1, 2, 3, \dots \quad (5)$$

where  $n$  is the instanton number. In the  $n$ -instanton sector, the integration variables in (2) are the classical solutions plus fluctuations,

$$x_\mu(\tau) = x_{0\mu}^{(n)}(\tau) + \delta x_\mu(\tau), \quad T = T_0^{(n)} + \delta T \quad (6)$$

and we shall expand the action in the fluctuations.

Before we proceed, we note that, in order to make the quadratic truncation of this theory well-defined, we need to deal with a collective coordinate. If  $x_{0\mu}^{(n)}(\tau)$  is a solution of the classical equation of motion, so is  $x_{0\mu}^{(n)}(\tau + t)$  and this degeneracy of solutions leads to a zero mode of the linearized equations of motion for the fluctuations. To fix this, we introduce a collective coordinate by inserting the identity

$$1 = \frac{1}{\omega} \int_0^1 dt \delta \left( \int_0^1 d\tau \left[ \dot{x}_{0\mu}^{(n)}(\tau) x_\mu(\tau + t) \right] \right) \left| \frac{d}{dt} \int_0^1 d\tau \dot{x}_{0\mu}^{(n)}(\tau) x_\mu(\tau + t) \right| \quad (7)$$

into the path integral. In this expression,  $\omega = n$  is the number of Gribov copies, that is, the number of solutions for the variable  $t$  of the equation  $\int_0^1 d\tau [\dot{x}_{0\mu}^{(n)}(\tau - t)x_\mu(\tau)] = 0$  in the interval  $t \in [0, 1)$ . (Notice that, being  $n$  cyclotron orbits,  $x_{0\mu}^{(n)}(\tau) = x_{0\mu}^{(n)}(\tau + 1/n)$ .) Upon using translation invariance of the integrand and measure, we can see that this is equivalent to inserting the gauge fixing factor

$$\int [dbdc\bar{c}] e^{i \int_0^1 d\tau [2\pi\omega b \dot{x}_{0\mu}^{(n)}(\tau)x_\mu(\tau) + \bar{c}c \dot{x}_{0\mu}^{(n)}(\tau)\dot{x}_\mu(\tau)]} \quad (8)$$

into the path integral. Here, the  $\tau$ -independent variables  $c$  and  $\bar{c}$  are anti-commuting Faddeev-Popov ghosts and  $b$  is a Lagrange multiplier.

We add the exponent in the integrand of equation (8) to the classical action  $S$  from equation (3) to get the “gauge fixed action”  $S_{\text{gf}}$ . Taking the classical parts of  $b$ ,  $c$  and  $\bar{c}$  as vanishing and of  $x_\mu(\tau)$  and  $T$  as before is consistent with the equations of motion obtained from  $S_{\text{gf}}$ . The world-line path integral is now

$$i\gamma = -2 \sum_{n=1}^{\infty} \frac{1}{V} \int [d(\delta x_\mu(\tau))d(\delta T)dbdc\bar{c}] \frac{1}{T_0^{(n)} + \delta T} e^{-S_{\text{gf}}[x_{0\mu}^{(n)} + \delta x_\mu, T_0^{(n)} + \delta T, b, c, \bar{c}]} \quad (9)$$

where, because it turns out that each instanton sector has an odd number of tachyons, the Euclidean partition function is purely imaginary in all of the instanton sectors (so we have removed the symbol  $\Im$ ). When it is expanded about the  $n$ -instanton solution, the action contains classical, quadratic and higher order (interaction) terms,

$$S_{\text{gf}} = S_{\text{classical}}^{(n)} + S_{\text{quad}}^{(n)} + S_{\text{int}}^{(n)} \quad (10)$$

respectively, and

$$S_{\text{classical}}^{(n)} = \int_0^1 d\tau \left[ \frac{T_0^{(n)}}{4} \dot{x}_{0\mu}^{(n)}(\tau)^2 + E x_{01}^{(n)}(\tau) \dot{x}_{02}^{(n)}(\tau) + \frac{m^2}{T_0^{(n)}} \right] = \frac{\pi n m^2}{E} \quad (11)$$

$$S_{\text{quad}}^{(n)} = \int_0^1 d\tau \left[ \frac{\delta T}{2} \dot{x}_{0\mu}^{(n)} \delta \dot{x}_\mu + \frac{T_0^{(n)}}{4} \delta \dot{x}_\mu^2 + E \delta x_1 \delta \dot{x}_2 + \frac{m^2 \delta T^2}{T_0^{(n)3}} + 2\pi\omega b \dot{x}_{0\mu}^{(n)} \delta x_\mu + \bar{c} c \dot{x}_{0\mu}^{(n)} \dot{x}_{0\mu}^{(n)} \right] \quad (12)$$

$$S_{\text{int}}^{(n)} = \int_0^1 d\tau \left[ \frac{1}{4} \delta T \delta \dot{x}_\mu(\tau) \delta \dot{x}_\mu(\tau) + \dot{x}_{0\mu}^{(n)}(\tau) \bar{c} c \delta \dot{x}_\mu(\tau) \right] + \sum_{k=3}^{\infty} \frac{m^2}{T_0^{(n)}} \frac{(-\delta T)^k}{T_0^{(n)k}} \quad (13)$$

It is in the remaining integral that we want to show that the interaction terms,  $S_{\text{int}}$ , can be replaced by zero and the term in the measure,  $1/T = 1/(T_0^{(n)} + \delta T)$ , can be replaced by  $1/T_0^{(n)}$ .

For this purpose, we define the fermionic transformation<sup>1</sup>

$$\Delta \bar{c} = \frac{1}{2} \delta T, \quad \Delta(\delta x_\mu(\tau)) = -c x_{0\mu}^{(n)}(\tau), \quad \Delta c = 0, \quad \Delta b = 0, \quad \Delta(\delta T) = 0 \quad (14)$$

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<sup>1</sup>This is not the BRST transformation associated with the gauge fixing.

It has the properties

$$\Delta^2(\text{anything}) = 0 \quad , \quad \int [d(\delta x_\mu(\tau))d(\delta T)dbdcd\bar{c}] \Delta(\text{anything}) = 0 \quad (15)$$

and the three parts of the action are invariant separately,

$$\Delta S_{\text{classical}}^{(n)} = 0 \quad , \quad \Delta S_{\text{quad}}^{(n)} = 0 \quad , \quad \Delta S_{\text{int}}^{(n)} = 0 \quad (16)$$

Moreover, the interaction terms in the action and in the measure can be seen to be exact, that is, they are

$$S_{\text{int}}^{(n)} = \Delta\psi \quad (17)$$

$$\psi = \frac{\bar{c}}{2} \int_0^1 d\tau \delta \dot{x}_\mu(\tau)^2 - 2\bar{c} \sum_{k=3}^{\infty} \frac{m^2}{T_0^{(n)}} \frac{(-\delta T)^{k-1}}{T_0^{(n)k}} \quad (18)$$

$$\frac{1}{T_0^{(n)} + \beta\delta T} = \frac{1}{T_0^{(n)}} + \Delta\chi \quad (19)$$

$$\chi = \bar{c} \sum_{k=1}^{\infty} \frac{(-\beta)^k \delta T^{k-1}}{T_0^{(n)k}} \quad (20)$$

Now, consider

$$\mathcal{I}(\beta, \lambda) \equiv \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \left( \frac{1}{T_0^{(n)} + \beta\delta T} \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda S_{\text{int}}^{(n)}}$$

We are interested in this integral when the parameters  $\beta = 1$  and  $\lambda = 1$ . However, we can easily show that it is independent of both  $\beta$  and  $\lambda$ . Consider

$$\begin{aligned} \frac{\partial}{\partial \lambda} \mathcal{I}(\beta, \lambda) &= \frac{d}{d\lambda} \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \left( \frac{1}{T_0^{(n)}} + \Delta\chi \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda \Delta\psi} \\ &= - \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \left\{ \left( \frac{1}{T_0^{(n)}} + \Delta\chi \right) \right\} (\Delta\psi) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda \Delta\psi} \\ &= - \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \Delta \left\{ \left( \frac{1}{T_0^{(n)}} + \Delta\chi \right) \psi e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda \Delta\psi} \right\} = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial \beta} \mathcal{I}(\beta, \lambda) &= \frac{\partial}{\partial \beta} \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \left( \frac{1}{T_0^{(n)}} + \Delta\chi \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda S_{\text{int}}^{(n)}} \\ &= \int [d(\delta x_\mu)d(\delta T)dbdcd\bar{c}] \Delta \left( \frac{d}{d\beta} \chi \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda S_{\text{int}}^{(n)}} \end{aligned}$$

$$= \int [d(\delta x_\mu) d(\delta T) dbdc d\bar{c}] \Delta \left\{ \left( \frac{d}{d\beta} \chi \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda S_{\text{int}}^{(n)}} \right\} = 0$$

and the integral

$$\mathcal{I}(\beta, \lambda) = \int [d(\delta x_\mu) d(\delta T) dbdc d\bar{c}] \left( \frac{1}{T_0^{(n)} + \beta \delta T} \right) e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)} - \lambda S_{\text{int}}^{(n)}}$$

is independent of the parameters  $\lambda$  and  $\beta$ . Both of these parameters can then be deformed to zero, yielding

$$\gamma = -\frac{2}{i} \sum_{n=1}^{\infty} \frac{1}{V} \int [d(\delta x_\mu(\tau)) d(\delta T) dbdc d\bar{c}] \frac{1}{T_0^{(n)}} e^{-S_{\text{classical}}^{(n)} - S_{\text{quad}}^{(n)}} \quad (21)$$

which, in each instanton sector, contains the classical solution plus quadratic fluctuations only. The remaining integral was performed in reference [5] where zeta-function regularization was used to define the infinite sums and products which appear. The result yields the known Schwinger formula (1).

In a similar vein, it is possible to show that imaginary part of the sigma model functional integral for oriented open strings in a constant electric field is given exactly by the WKB limit of the semi-classical expansion about world-sheet instantons, and summing over instanton number [6].

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